UDC 539.3

NONAXISYMMETRIC PROBLEMS OF SHALLOW SHELLS OF REVOLUTION UNDER FINITE DISPLACEMENTS

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Questions of the passage of axisymmetric into nonaxisymmetric equilibrium modes are investigated for shallow shells of revolution. Diverse modifications of the method of factorization are analyzed in application to the problems under consideration. Results of computations for spherical and conical shells are presented as an illustration.

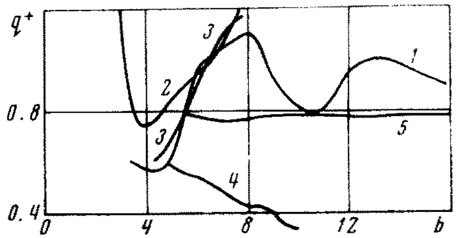

Fig. 1

The investigation of the nonaxisymmetric stability problem of shallow spherical shells of revolution with rigidly clamped supporting contour, loaded by an external hydrostatic pressure was started in [1] by using the Bubnov-Galerkin method. Later analogous investigations were carried out by energy methods in [2, 3](*). Corresponding results of the dependences of the critical loads q^+ on the shallowness parameter b are represented in Fig. 1 by curves 2 and 3. Curve 1 determines the parameter of the critical load obtained by solving the axisymmetric problem.

The first investigation of the problem by a numerical method was carried out in [4]. In contrast to [1 – 3], it was considered that the shell equilibrium mode is axisymmetric in the first stage, and the uniqueness condition is violated when the load reaches some limiting level, and then nonaxisymmetric equilibrium modes appear together with the axisymmetric modes. The results obtained are represented by curve 4 in Fig. 1. Analogous investigations were later carried out by finite differences (curve 5) in [5]. An iteration algorithm [6] was hence used to determine the principal state of stress. Despite the identical formulations of the problem and the equivalence of the governing equations, the final results of [4, 5] differ substantially. They do not even agree with the results in [2, 3]. Analogous investigations were carried out on the basis of another numerical algorithm in [7]. The results in [5, 7] agree to 6% accuracy for small values of b; they are examined in [7].

Shells with different edge conditions for the supporting contour were investigated in [8]. The axisymmetric problem was solved on the basis of a numerical

*) References to [3, 4] and their results are borrowed from [5].

algorithm providing for the reduction of the nonlinear boundary value problem to a system of nonlinear algebraic equations [9]. The equations determining the nonaxisymmetric solutions were integrated numerically by the initial parameters method. Values of the critical loads for shells with rigidly clamped supporting contour, obtained in [5, 8], do not agree. The difference between the results of solving the nonaxisymmetric problems, which does not exceed 2 - 3%, is explained by the diverse values of the Poisson's ratio. The difference in the critical load values q^+ for the nonaxisymmetric problems, which refers primarily to shells with large values of the parameter b, is more essential. Thus, for example, it reaches 4.2% for $b = 11$. This is explained by the fact that for large b the algorithm in [5] does not permit exact determination of the position of the limit points. Hence, values of the critical load are presented to the accuracy of the first two significant figures for $b \geqslant 9$ in the paper mentioned. Similar difficulties are characteristic for many existing algorithms. For example, convergent processes could not be constructed successfully for $b \geqslant b_0$ in [10]. The limit value b_0 depends essentially on the nature of the conditions of the supporting contour. The algorithm of [9] has no such disadvantage.

The investigation of nonaxisymmetric problems is of interest precisely for shells with large values of b. For such shells the solution of the axisymmetric problems, whose results are used directly in the analysis of the nonaxisymmetric problems, is fraught with considerable difficulties. They are overcome in this paper by using the algorithm of [9] in combination with the method of dividing the segment of integration into intermediate segments [10]. The difficulties associated with the rapid growth of the solution for the nonaxisymmetric problems are overcome by using diverse factorization methods [11 – 15].

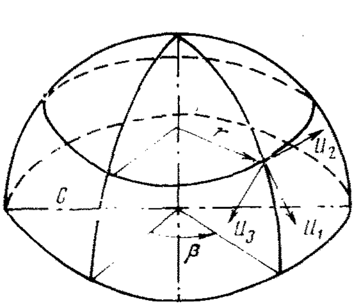

Fig. 2

Let us consider an arbitrary shallow shell of revolution with variable thickness h along the meridian, whose temperature t varies along the thickness and along the meridian. The material characteristics E_i, G, ν_i, α_i, the elastic moduli of the first and second kinds, the Poisson's ratio and coefficient of linear expansion in the meridian $(i = 1)$ and circumferential $(i = 2)$ directions, vary along the meridian.

The strains and the changes in curvature are expressed in terms of the displacement vector component u (see Fig. 2) by using the relationships [14]

$$\varepsilon_{11} = \frac{\partial u_1}{\partial r} - \frac{u_3}{R_1} + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2, \quad \varepsilon_{22} = \frac{\partial u_2}{r\partial\beta} + \frac{u_1}{r} - \frac{u_3}{R_2} + \frac{1}{2}\left(\frac{\partial u_3}{r\partial\beta}\right)^2 \quad (1)$$

$$\varepsilon_{12} = \frac{\partial u_1}{r\partial\beta} + r\frac{\partial}{\partial r}\left(\frac{u_2}{r}\right) + \frac{\partial u_3}{r\partial\beta}\frac{\partial u_3}{\partial r}, \quad H_{11} = -\frac{\partial^2 u_3}{\partial r^2}$$

$$H_{22} = -\frac{\partial^2 u_3}{r^2\partial\beta^2} - \frac{\partial u_3}{r\partial r}, \quad H_{12} = \frac{\partial u_3}{r^2\partial\beta} - \frac{\partial^2 u_3}{r\partial r\partial\beta}$$

Expressions for the stress resultants and moments

$$N_{11} = \frac{E_1 h}{1-\nu_1\nu_2}(\varepsilon_{11}+\nu_2\varepsilon_{22}) - \frac{E_1}{1-\nu_1\nu_2}(\beta_1+\nu_2\beta_2), \quad N_{12} = Gh\varepsilon_{12} \tag{2}$$

$$N_{22} = \frac{E_2 h}{1-\nu_1\nu_2}(\varepsilon_{22}+\nu_1\varepsilon_{11}) - \frac{E_2}{1-\nu_1\nu_2}(\beta_2+\nu_1\beta_1), \quad M_{12} = \frac{Gh^3}{6} H_{12}$$

$$M_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}(H_{11}+\nu_2 H_{22}) - \frac{E_1}{1-\nu_1\nu_2}(\gamma_1+\nu_2\gamma_2)$$

$$M_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}(H_{22}+\nu_1 H_{11}) - \frac{E_2}{1-\nu_1\nu_2}(\gamma_2+\nu_1\gamma_1)$$

$$\beta_i = \alpha_i \int_{-h/2}^{h/2} t\,dz, \qquad \gamma_i = \alpha_i \int_{-h/2}^{h/2} tz\,dz$$

follow from the general statements of shell theory and the linear elasticity relationships.

Let us write the equilibrium equations neglecting transverse forces in the first two of them, as is customary for shallow shells [14]

$$\frac{\partial(rN_{11})}{r\partial r} + \frac{\partial N_{12}}{r\partial\beta} - \frac{N_{22}}{r} - \frac{\gamma h}{g}\frac{\partial^2 u_1}{\partial\tau^2} = 0 \tag{3}$$

$$\frac{\partial N_{22}}{r\partial\beta} + \frac{\partial(rN_{12})}{r\partial r} + \frac{N_{12}}{r} - \frac{\gamma h}{g}\frac{\partial^2 u_2}{\partial\tau^2} = 0$$

$$\frac{\partial(rQ_1)}{r\partial r} + \frac{\partial Q_2}{r\partial\beta} + \left(\frac{1}{R_1} - H_{11}\right)N_{11} + \left(\frac{1}{R_2} - H_{22}\right)N_{22} -$$

$$2H_{12}N_{12} + q - \frac{\gamma h}{g}\frac{\partial^2 u_3}{\partial\tau^2} = 0$$

$$\frac{\partial(rM_{11})}{r\partial r} + \frac{\partial M_{12}}{r\partial\beta} - \frac{M_{22}}{r} - Q_1 = 0, \quad \frac{\partial(rM_{12})}{r\partial r} + \frac{\partial M_{22}}{r\partial\beta} + \frac{M_{12}}{r} - Q_2 = 0$$

We introduce the following computational parameters

$$M_{ij}^* = \eta\frac{M_{ij}R_0}{E_0 h_0}, \quad N_{ij}^* = \sqrt{\eta}\,\frac{N_{ij}R_0}{E_0 h_0^2}, \quad u_i^* = \eta^{3/4}\sqrt{\frac{R_0}{h_0}}\,\frac{u_i}{h_0}, \tag{4}$$

$$u_3^* = \sqrt{\eta}\,\frac{u_3}{h_0}$$

$$Q_i^* = \eta^{3/4}\sqrt{\frac{h_0}{R_0}}\,\frac{Q_i R_0^2}{E_0 h_0^3}, \quad \varepsilon_{ij}^* = \sqrt{\eta}\,\frac{R_0}{h_0}\varepsilon_{ij}, \quad H_{ij}^* = R_0 H_{ij}, \; r^* = b\frac{r}{c}$$

$$q^* = \frac{\sqrt{\eta}}{4}\,\frac{q}{E_0}\left(\frac{R_0}{h_0}\right)^2, \quad \tau^* = \sqrt{\frac{E_0 g}{\gamma R_0^2}}\,\tau, \quad \lambda_i = \frac{R_0}{R_i}, \quad b = \sqrt[4]{\eta}\,\frac{C}{\sqrt{R_0 h_0}}$$

$$\beta_i^* = \sqrt{\eta}\,\frac{R_0}{hh_0}\beta_i, \quad \gamma_i^* = 12\frac{R_0}{h^3}\gamma_i, \quad a_{i1} = \frac{E_i h}{E_0 h_0}$$

$$a_{i3} = \frac{E_i h^3}{E_0 h_0^3}, \quad a_1 = \frac{Gh}{E_0 h_0}, \quad a_3 = \frac{Gh^3}{E_0 h_0^3}$$

$$\eta = 12(1-\nu_1\nu_2)$$

where E_0, h_0 are the characteristic elastic modulus and thickness, respectively, R_0 is the radius of curvature of the supporting contour in the circumferential direction, and c is the radius of the supporting contour. (The asterisk will henceforth be omitted in the notation).

Let us transform (1) – (3) by taking account of (4). We hence obtain

$$\left(\frac{u_1}{r}\right)' = -\frac{1+\nu_2}{r}\frac{u_1}{r} - \frac{\nu_2}{r}\left(\frac{u_2}{r}\right)^{\cdot} + \frac{\lambda_1+\nu_2\lambda_2}{r}u_3 - \tag{5}$$
$$\frac{\theta^2}{2r} + \frac{1-\nu_1\nu_2}{ra_{11}}N_{11} - \frac{\nu_2}{2r^3}u_3^{\cdot 2} + \frac{\beta_1+\nu_2\beta_2}{r}$$
$$\left(\frac{u_2}{r}\right)' = -\frac{1}{r}\left(\frac{u_1}{r}\right)^{\cdot} - \frac{\theta}{r^2}u_3^{\cdot} + \frac{N_{12}}{a_1 r}$$
$$\theta' = -\frac{\nu_2}{r^2}u_3^{\cdot\cdot} - \frac{\nu_2}{r}\theta - \frac{M_{11}}{a_{23}} - (\gamma_1+\nu_2\gamma_2), \quad u_3' = \theta$$
$$N_{11}' = \frac{a_{21}}{r}\frac{u_1}{r} + \frac{a_{21}}{r}\left(\frac{u_2}{r}\right)^{\cdot} - \frac{a_{21}\lambda_2}{r}u_3 - \frac{1-\nu_2}{r}N_{11} -$$
$$\frac{N_{12}^{\cdot}}{r} + \frac{a_{21}u_3^{\cdot 2}}{2r^3} - \frac{a_{21}\beta_2}{r} - \frac{h}{\sqrt[4]{\eta}R_0}\frac{\partial^2 u_1}{\partial\tau^2}$$
$$N_{12}' = -\frac{a_{21}}{r}\left(\frac{u_1}{r}\right)^{\cdot} + \frac{a_{21}\lambda_2}{r}u_3^{\cdot} -$$
$$\frac{a_{21}}{r^2}u_3^{\cdot}u_3^{\cdot\cdot} - \frac{\nu_2}{r}N_{11}^{\cdot} - \frac{2}{r}N_{12} + \frac{h}{\sqrt[4]{\eta}R_0}\frac{\partial^2 u_2}{\partial\tau^2}$$
$$S' = -\frac{S}{r} - \frac{\nu_2}{r^2}M_{22}^{\cdot\cdot} + \frac{1-\nu_1\nu_2}{r^4}(a_{23}u_3^{\cdot\cdot\cdot\cdot} - 4a_3u_3^{\cdot\cdot}) +$$
$$\frac{1-\nu_1\nu_2}{r^3}(a_{23}+4a_3)\theta^{\cdot\cdot} - \left(\lambda_1 - \frac{\nu_2}{r^2}u_3^{\cdot\cdot} -\right.$$
$$\left.\frac{\nu_2}{r}\theta - \frac{M_{11}}{a_{13}} - \gamma_1 - \nu_2\gamma_2\right)N_{11} - \left(\lambda_2 + \frac{u_3^{\cdot\cdot}}{r^2} + \frac{\theta}{r}\right)\times$$
$$\left[\nu_2 N_{11} + a_{21}\left(\frac{u_2}{r}\right)^{\cdot} + a_{21}\frac{u_1}{r} - a_{21}\lambda_2 u_3 + \frac{a_{21}}{2r^2}u_3^{\cdot 2}\right] +$$
$$\frac{h}{h_0}\frac{\partial^2 u_3}{\partial\tau^2} - 4q, \quad M_1' = \frac{1-\nu_1\nu_2}{r^2}(4a_3\theta^{\cdot\cdot} - a_{23}\theta) - \frac{M_{11}}{r} -$$
$$\frac{1-\nu_1\nu_2}{r^3}(a_{23}+4a_3)u_3^{\cdot\cdot} - \frac{1-\nu_1\nu_2}{r}a_{23}\gamma_2, \quad S = Q_1 + \frac{M_{12}^{\cdot}}{r}$$

The prime here denotes the derivative with respect to r, and the dot the derivative with respect to β.

An axisymmetric equilibrium mode corresponds to the initial unperturbed state. The functions governing it depend only on r, are denoted by the zero superscript, and are determined by the solution of the system (5) which it is expedient to reduce to the form

$$\varepsilon_{22}^{\circ\prime} = -\frac{1+\nu_2}{r}\varepsilon_{22}^{\circ} - \left(\lambda_2 + \frac{\theta^{\circ}}{2r}\right)\theta^{\circ} + \frac{1-\nu_1\nu}{a_{11}r}N_{11}^{\circ} + \frac{\beta_1+\nu_2\beta_2}{r} \tag{6}$$
$$N_{11}^{\circ\prime} = \frac{a_{21}}{r}\varepsilon_{22}^{\circ} - \frac{1-\nu_2}{r}N_{11}^{\circ} - \frac{a_{21}}{r}\beta_2, \quad \theta^{\circ\prime} = -\frac{\nu_2}{r}\theta^{\circ} - \frac{M_{11}^{\circ}}{a_{23}} -$$
$$\gamma_1 - \nu_2\gamma_2$$
$$M_{11}^{\circ\prime} = -\frac{1-\nu_1\nu_2}{r^2}a_{23}\theta^{\circ} + Q_1^{\circ} - \frac{1-\nu_1\nu_2}{r}a_{23}\gamma_2$$
$$Q_1^{\circ} = -r\left(\lambda_2 + \frac{\theta_0}{r}\right)N_{11}^{\circ} - \frac{4}{r}\int_0^r qr\,dr + \frac{A_0}{r}$$

The constant A_0 is determined from the equilibrium condition for the central part of the shell. In particular, for an external hydrostatic pressure $q = \text{const}$

$$Q_1^\circ = -r\left(\lambda_2 + \frac{\theta^\circ}{r}\right)N_{11}^\circ - 2qr$$

The perturbed state characterized by small deviations from the initial equilibrium mode is representable as

$$\frac{u_1}{r} = \frac{u_1^\circ}{r} + x_1, \quad \frac{u_2}{r} = x_2, \quad u_3 = u_3^\circ + x_3, \quad \theta = \theta^\circ + x_4 \tag{7}$$

$$N_{11} = N_{11}^\circ + x_5, \quad N_{12} = x_6, \quad S = Q_1^\circ + x_7, \quad M_{11} = M_{11}^\circ + x_8$$

Substituting (7) into the system (5) and linearizing, taking account of the smallness of x_s, results in a system of equations whose solution is

$$x_s = \sum_{k=0}^{\infty} x_s^k e^{i\omega_k\tau} \sin k\beta \qquad (s \neq 2,6)$$

$$x_s = \sum_{k=0}^{\infty} x_s^k e^{i\omega_k\tau} \cos k\beta \qquad (s = 2,6)$$

As a result of manipulations, we obtain a system of equations for each number k (we omit the subscript in the notation)

$$x_1' = -\frac{1+\nu_2}{r}x_1 + \frac{\nu_2 k}{r}x_2 + \frac{\lambda_1 + \nu_2\lambda_2}{r}x_3 - \frac{\theta^\circ}{r}x_4 + \frac{1-\nu_1\nu_2}{a_{11}r}x_5 \tag{8}$$

$$x_2' = -\frac{k}{r}x_1 - \frac{k\theta^\circ}{r^2}x_3 + \frac{x_6}{ra_1}, \quad x_3' = x_4, \quad x_4' = \frac{k^2\nu_2}{r^2}x_3 - \frac{\nu_2}{r}x_4 - \frac{x_8}{a_{23}}$$

$$x_5' = \frac{a_{21}}{r}x_1 - \frac{ka_{21}}{r}x_2 - \frac{a_{21}\lambda_2}{r}x_3 - \frac{1-\nu_2}{r}x_5 + \frac{k}{r}x_6 - \frac{hr\omega_k^2}{\sqrt[4]{\eta}R_0}x_1$$

$$x_6' = -\frac{a_{21}k}{r}x_1 + \frac{a_{21}k^2}{r}x_2 + \frac{\lambda_2 a_{21}k}{r}x_3 - \frac{\nu_2 k}{r}x_5 - \frac{2}{r}x_6 - \frac{hr\omega^2 k}{\sqrt[4]{\eta}R_0}x_2$$

$$x_7' = -a_{21}\left(\lambda_2 + \frac{\theta^\circ}{r}\right)x_1 + a_{21}k\left(\lambda_2 + \frac{\theta^\circ}{r}\right)x_2 + \left[\frac{1-\nu_1\nu_2}{r^4}(k^2a_{23} + 4a_3)k^2 + a_{21}\lambda_2\left(\lambda_2 + \frac{\theta^\circ}{r}\right) + \frac{a_{21}k^2\varepsilon_2^0}{r^2} - \frac{h}{h_0}\omega_k^2\right]x_3 - \left[\frac{1-\nu_1\nu_2}{r^3}(a_{23} + 4a_3)k^2 + \frac{a_{23}\varepsilon_2^\circ}{r}\right]x_4 + \left(\frac{M_{11}}{a_{13}} - \lambda_1 - \nu_2\lambda_2 + \gamma_1 + \nu_2\gamma_2\right)x_5 - \frac{x_7}{r} + \left(\frac{N_{11}^\circ}{a_{13}} + \frac{\nu_2k^2}{r^2}\right)x_8$$

$$x_8' = \frac{1-\nu_1\nu_2}{r^3}(a_{23} + 4a_3)k^2x_3 - \frac{1-\nu_1\nu_2}{r^2} \times (a_{23} + 4k^2a_3)x_4 + x_7 - \frac{1-\nu_2}{r}x_8$$

Two exceptional cases should be considered separately. For $k = 0$ we have $x_2 = x_6 = 0$. If only normal components of the inertial forces are taken into account, then

$x_6 = -x_5$ holds for $k = 1$. If inertial forces are not taken into account at all, the stability problem, for example, is solved in the static formulation, then for $k = 1$ the seventh equation of the system can be reduced to the form

$$x_7 = -\frac{N_{11}^\circ}{r}x_3 - N_{11}^\circ x_4 - r\left(\lambda_2 + \frac{\theta^\circ}{r}\right)x_5 + \frac{x_8}{r} \tag{9}$$

The order of the system is reduced in the cases noted. In the general case $(k \geqslant 2)$ the system (8) can be written as

$$\mathbf{X}' = \mathbf{A}X \tag{10}$$

where X is a column-vector with coordinates x_s and A is a square matrix of dimensionality 8×8.

The problem on the passage of the axisymmetric equilibrium modes into nonaxisymmetric modes consists of determining the matrix A for which the system (10) has a nontrivial as well as trivial solution. The matrix itself depends on the load parameter, the temperature and the frequency. Therefore, the problem of determining the eigennumbers and eigenvectors of the linear differential operator (10) is solved in substance.

In carrying out the investigations let us first solve the nonlinear boundary value problem for the system (6) by using the algorithm [9]. Difficulties originate for shells with large values of the parameter b because of the presence of growing solutions. Hence, the algorithm is used in combination with the method of dividing the integration segment into intermediate segments.

The integration segment $[\Delta, b]$ was divided into four parts for the numerical realization of the algorithm. Continuity conditions for the governing functions at the division points and the boundary conditions were hence used. Consequently, the nonlinear boundary value problem was reduced to a system of 14 nonlinear algebraic equations with the same number of unknowns. The results obtained were used in parallel integration of the system (6) and (10) to establish the critical values of the parameters.

Let us examine different methods of integrating the system (10) applied to shells closed at the vertex. For unclosed shells the method is altered insignificantly because of the other initial conditions.

Integration of (10) can be accomplished by the method of initial parameters. To do this we represent the vector $\mathbf{X}$ and the matrix A in the form

$$\mathbf{X} = \begin{Vmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{Vmatrix}, \qquad A = \begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix} \tag{11}$$

Here $\mathbf{X}_1$ is the displacement vector, $\mathbf{X}_2$ is the stress resultant vector, and A_{ij} are matrices of rank 4×4.

For $r = 0$ there is a singularity, hence, integration of the system (10) is realizable from a point close to the vertex $r = \Delta$. In the eighborhood of $r \in [0, \Delta]$ we have $\mathbf{X}_1 \approx 0$. Hence, we obtain four linearly independent solutions of the system by taking the following initial values of the vectors (12)

$$\mathbf{X}_1^n(\Delta) = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \quad \mathbf{X}_2^1(\Delta) = \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \quad \mathbf{X}_2^2(\Delta) = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}, \quad \mathbf{X}_2^3(\Delta) = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Vmatrix}, \quad \mathbf{X}_2^4(\Delta) = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Vmatrix}$$

$(n = 1, 2, 3, 4)$

The general solution of the system is

$$\mathbf{X} = \sum_{n=1}^{4} C_n \mathbf{X}^n$$

From the boundary condition of the supporting contour we obtain four linear homogeneous algebraic equations in C_n for $r = b$ which for shells with rigidly clamped supporting contour are of the form

$$\sum_{n=1}^{4} C_n \mathbf{X}_1{}^n (b) = 0 \tag{13}$$

The system (12) has a nontrivial solution if the condition

$$| \mathbf{X}_1{}^1 (b) \quad \mathbf{X}_1{}^2 (b) \quad \mathbf{X}_1{}^3 (b) \quad \mathbf{X}_1{}^4 (b) | = 0$$

is satisfied. This condition determines the critical values of the load, the temperatures or the frequency of the natural system vibrations for a shell with rigidly clamped supporting contour. For other kinds of support of the contour, the condition is formualted analogously.

Therefore, a solution of the problem for spherical and conical shells has been obtained successfully for $b \leqslant 10 - 12$ and $k \leqslant 10 - 12$. For large b difficulties arise in the integration because of the presence of growing solutions and, what is more serious, because the vectors $\mathbf{X}^n$ become almost linearly dependent during the integration. These difficulties are overcome by partitioning the integration segment into m intermediate segments, by orthonormalizing the vectors obtained at these points, and by using the results as initial values to continue the integration into the next segment [11].

As b and k increase, m grows abruptly. Thus, it is sufficient to take $m = 5 - 10$ for $b = 18$ and $k = 13 - 18$, while we need $m = 160 - 320$ for $b = 42$ and $k = 35 - 50$. For an extreme increase in m there is the danger of a loss in accuracy. Hence, verification of the results by other methods is necessary.

Because of the linearity of the problem the condition (the asterisk denotes the transpose)

$$B^* \mathbf{X} = 0 \tag{14}$$

is satisfied for the stress resultant and displacement vectors. Here B^* is a rectangular matrix of the dimensionality 8×4. For the transpose matrix B the differential equation is [13]

$$B' = B\,(B^*B)^{-1} B^* A^* B - A^* B \tag{15}$$

It can be proved that

$$B^* B = \text{const} \tag{16}$$

Condition (16) indicates the expediency of integrating (15) instead of the initial system (10). For shells closed at the vertex, by virtue of (12), we have

$$B^* (\Lambda) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix} \tag{17}$$

If (16) and (17) are taken into account, then (15) is written as

$$B' = BB^* A^* B - A^* B \tag{18}$$

Therefore, we have the initial condition (17) for the matrix equation (18). To formulate the conditions on the supporting contour, let us represent the matrix B^* as

$$B^* = (B_1^* B_2^*)$$

where B_1^* and B_2^* are square matrices of dimensionality 4×4. It follows from the relationships (11) and (14) that

$$B_1^* \mathbf{X}_1 + B_2^* \mathbf{X}_2 = 0$$

The condition for the existence of a nontrivial solution is written as $B_2^*(b) = \vartheta$ for shells with a rigidly clamped contour. The conditions for the other kinds of contour support are analogous (difficulties in realization of the algorithm do not arise). The question of finding the eigenvectors corresponding to the eigennumbers found for the operator (10) is solved simply and is hence not discussed.

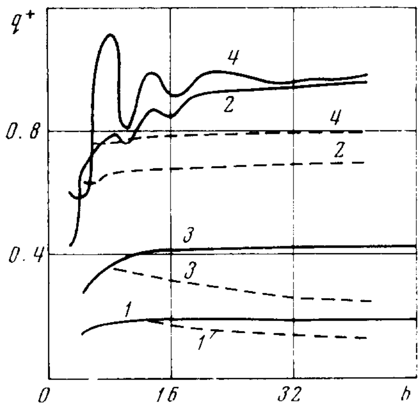

Fig. 3

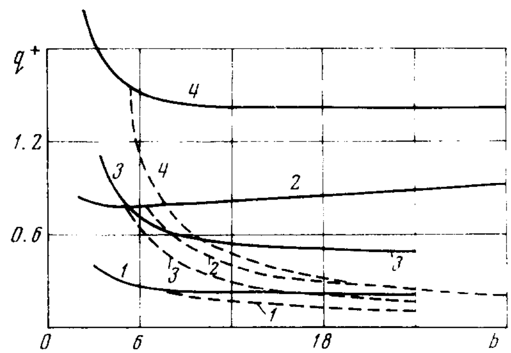

Fig. 4

Table 1

b	Number of the Problem							
	1		2		3		4	
	k	$q^+\cdot 10^3$	k	$q^+\cdot 10^3$	k	$q^+\cdot 10^3$	k	$q^+\cdot 10^3$
4	0	137	0	663	0	—	0	564
6	0	165	0	743	0	311	0	972
6			2	651			2	764
6			3	631			3	819
6			4	675			4	920
10	0	179	0	752	0	386	0	811
10			5	676	3	344	4	804
10			6	660	4	338	5	768
10			7	672	5	351	6	772
18	0	187	0	898	0	413	0	920
18	6	165	11	687	8	301	11	791
18	7	163	12	681	9	301	12	785
18	8	164	13	682	10	302	13	786
30	0	192	0	940	0	425	0	951
30	11	143	22	691	14	261	21	794
30	12	143	23	691	15	261	22	793
30	13	144	24	693	16	263	23	793
42	0	193	0	958	0	430	0	968
42	15	127	29	797	16	244	30	803
42	16	127	30	695	17	241	31	798
42	17	129			18	241	32	798

The problem of passing from the axisymmetric to the nonaxisymmetric equilibrium modes has been examined for isotropic spherical and conical constant-thickness shells subjected to a constant-intensity external pressure. Graphs of the dependence of the critical load parameter q^+ on the shallowness parameter b are represented in Fig. 3 for spherical and in Fig. 4 for conical shells. The continuous lines have been constructed from the results of solving the axisymmetric problem, while the dashes determine the value of the minimal critical load corresponding to the passage from axisymmetric over to nonaxisymmetric equilibrium modes. For shells with different boundary conditions for the supporting contour the following numbering is taken: moving hinge is Problem 1, fixed hinge is Problem 2, moving clamping is Problem 3, and rigid clamping is Problem 4. The numerical values of q^+ for spherical shells are presented in Table 1.

For small values of the shallowness parameter $b < b_0$ up to buckling, axisymmetric equilibrium modes are realized. However, for $b \geqslant b_0$ the axisymmetric equilibrium modes are not unique and nonaxisymmetric modes are possible. The limit value of the parameter $b = b_0$ depends on the geometry of the middle surface and the nature of the boundary of the shell supporting contour. This is easy to conceive if it is taken into account that the reason for the displacement of the equilibrium mode is the high level of the circumferential compressive stress resultants at the supporting contour, which depends primarily on the factors mentioned above. The nature of fixing the supporting contour strongly affects the value of the critical load.

As the parameter b increases for a spherical shell with a moving supporting contour (Problems 1 and 3), the critical load q^+ diminishes approximately 1.5-fold for large b Another picture is observed for shells with a fixed supporting contour (Problems 2 and 4) for which the critical load is practically constant for large b. For such shells the solution of the axisymmetric problem yields an exaggerated value (approximately 1.5-fold) of the critical load.

The picture observed for conical shells is different. As the parameter b increases the critical load diminishes (Fig. 4) for all kinds of conditions considered for the supporting contour. Thus, for example, for a shell with a rigidly clamped supporting contour and a shallowness parameter $b = 30$ taking account of the nonaxisymmetric modes results in an approximately sevenfold reduction in q^+.

As b increases the number of the harmonic k_+ to which the minimum critical load corresponds also increases. However, an approximately constant value $l_+ = 2\pi b/k_+$ is

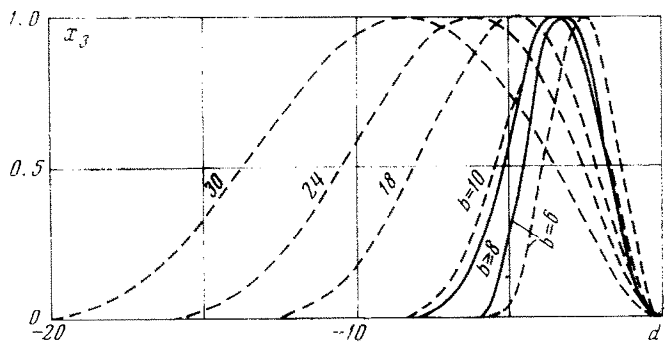

Fig. 5

established, i.e. some limit wavelength of the buckling mode exists for which the critical load is minimal. For large b close harmonics have approximately identical values of l_+, hence, the minimal critical load can be realized simultaneously for several harmonics.

The buckling modes of shells with a rigidly clamped supporting contour are presented in Fig. 5; by solid lines for spherical shells for $k = k_+$ and by dashes for conical shells (x_3 is the parameter of the nonaxisymmetric component of the normal displacement, and $d = r - b$). An analogous picture holds for shells with other boundary conditions for the supporting contour.

The equilibrium modes realized for minimal critical loads for spherical shells are distinct only for small values of b, while they practically agree for large values of b. They are independent of k_+ (as has already been remarked, the critical load can be realized for several numbers of the harmonics). The edge effect zone is slight with the exception of shells with a moving supporting contour.

A different picture is observed for conical shells for which the edge effect zone is large, as a rule, independently of the boundary condition for the supporting contour. If the minimal critical load hence holds for several values of k_+, then the corresponding equilibrium modes differ, but insignificantly.

Free vibrations of spherical shells under finite displacements have been investigated by numerical methods in [16].

REFERENCES

1. Grigoljuk, E.J., On the unsymmetrical snapping of shells of revolution. Proc. IUTAM Sympos. on Thin Elastic Shell Theory, Amsterdam, North Holland Publishing Co., Delft, 1960.
2. Gjelsvik, Bodner, Nonsymmetrical snap buckling of spherical caps. Proc. Amer. Soc. Civil Eng'rs., J. Eng'ng. Mech. Div., Vol. 88, № 5, 1962.
3. Parmerter, R.D. and Fung, Y.C., On the influence of nonsymmetrical modes on the buckling of a shallow spherical shell. NASA TN D-1510, 1962.
4. Weinitschke, H.J., Asymmetric buckling of clamped shallow spherical shells. NASA TN D-1510, 1962.
5. Huang, N.G., Unsymmetrical buckling of thin shallow spherical shells. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 31, № 3, 1964.
6. Budiansky, B., Buckling of clamped shallow spherical shells. Proc. Sympos. on Thin Elastic Shell Theory, Amsterdam, North Holland Publ. Co., Delft, 1959.
7. Shil'krut, D.I. and Gerlaku, I.D., Determination of the critical loads of axisymmetrically loaded shallow, geometrically nonlinear, shells of revolution taking into account nonsymmetric equilibrium modes. In: Questions of the Nonlinear Theory of Shells and Rods. Kishinev, 1969.
8. Valishvili, N.V., Nonsymmetric deformation and stability of shallow shells of revolution. In: Theory of Plates and Shells. "Nauka", Moscow, 1971.
9. Valishvili, N.V., An algorithm of the solution of nonlinear boundary value problems. PMM Vol. 32, № 6, 1968.
10. Weinitschke, H.J., On the stability problem for shallow spherical shells. J. Math. and Phys., Vol. 38, № 4, 1960.

11. Godunov, S. K., On the numerical solution of boundary value problems for systems of linear ordinary differential equations. Usp. Matem. Nauk, Vol. 16, №3, 1961.
12. Grigoliuk, E. I., Mal'tsev, V. P., Miachenkov, V. I. and Frolov, A. N., On a method of solving stability and vibrations problems for shells of revolution. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, №1, 1971.
13. Abramov, A. A., On the transfer of boundary conditions for systems of linear ordinary differential equations. (English translation), Pergamon Press, Zh. vychisl. Mat. i mat. Fiz., Vol. 1, №3, 1961.
14. Vol'mir, A. S., Stability of Deformable Systems. "Nauka", Moscow, 1967.
15. Weinitschke, H. J., Asymmetric buckling of shallow spherical shells. J. Math. and Phys., Vol. 44, №1, 1965.
16. Valishvili, N. V., Free Vibrations of Shallow Shells of Revolution under Finite Displacements. "Nauka", Moscow, 1973.

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STOCHASTIC STABILITY OF FORCED NONLINEAR SHELL VIBRATIONS

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The parameters of stationary forced nonlinear vibrations are determined within the framework of correlation theory for shells considered as a system with one degree of freedom and subjected to a transverse pressure which is random in time. The generalized force is described as a stationary normal process with a rational fraction spectral density.

The stability of the solutions found is verified by the perturbed motion equation in a linear approximation. The system is first reduced to a Markov type by extension of the phase space. Then the Liapunov theorem on stability in a linear approximation is applied to the set of first and second order moment functions. The final stage in the problem is executed by numerical methods.

It is disclosed that there are unstable solutions in some domain of the parameter space. Jump-like transitions from some stable states to others are observed for systems with comparatively large nonlinearity.

Characteristic kinds of deterministic loadings have been investigated in [1–4]. For essentially nonlinear systems the curves of the states have sections corresponding to unstable motions.

Stationary forced vibrations of shells under random loads have been examined in a number of papers [5–7]. Investigations conducted within the framework of the correlation approximation often yield ambiguous solutions and the question of what motions are realized, remains open.

The main purpose herein is to extract those of the solutions which correspond to the unstable vibrations, and thereby determine the actual shell behavior more accurately.